Erdos-Szekeres Theorem

1. Theorem:

Given n^2+1 real numbers, there is always an inc or dec sub-sequence consisting of $n+1$ numbers.

Sequence: An enumerated collection of objects in which order matters.
I.e. $1, 3, 5, 6 \neq 1, 5, 3, 6$

Sub-Sequence: What remains after we remove some terms from a sequence.
I.e. $1, 3$ is a sub-seq of $1, 3, 5, 6$

Increasing Sequence: A seq is inc if each term $\geq$ than the prev term.
I.e. $x_1 \leq x_2 \leq \ldots \leq x_n$

Decreasing Sequence: A seq is dec if each term is $\leq$ than the prev term.
I.e. $x_1 \geq x_2 \geq \ldots \geq x_n$

2. General Proof:

Consider the n^2+1 nums $a_1, a_2, \ldots, a_{n^2+1}$. Suppose that we want to find $n+1$ numbers s.t. they form a decreasing sub-sequence.
There are 2 cases:

Case 1: Consider the longest decreasing sub-seq starting from a_1. The max possible length the sub-seq can have is n^2+1.
I.e. $a_1 \geq a_2 \geq \ldots \geq a_{n^2+1}$
If there exists a sub-sequence with length $\geq n+1$, then we are done.

Case 2: Suppose that there is no sub-seq starting from a_i, $i \in \{1, 2, \ldots, n^2+1\}$ that has length $\geq n+1$.
I.e. Suppose all the sub-seqs have a length between 1 and n, inclusive.
Since there are n^2+1 sub-sequences and n different possible lengths, there will be $n+1$ sub-seq with the same length (By P.P.). Let these $n+1$ sub-seqs be $x_1, x_2, \ldots, x_{n+1}$ and suppose that they all have a length of n. Then, $x_1 \leq x_2 \leq \ldots \leq x_{n+1}$. Here is why: Consider x_1 and x_2 and suppose $x_1 > x_2$. We know that x_2 has a sub-seq of length n. If we were to add x_1 to that sub-seq, then the length would be $n+1$, violating our assumption that the lengths of the sub-seq is n.

I.e.: $\underbrace{x_1, \underbrace{x_2, \ldots,}_{n}}_{n+1}$

Hence $x_1 \leq x_2$. The same logic applies to all other x_i's.

3. Examples:

a) Prove that given 5 numbers, there must be 3 that form an inc or dec sub-seq.

Soln:
Let a_1, a_2, a_3, a_4, a_5 be the 5 numbers. Suppose we want to find a dec sub-seq.

Case 1:
Consider the longest dec sub-seq starting with a_1. The length is 5.

Consider the longest dec sub-seq starting with a_2. The length is 4.

Consider the longest dec sub-seq starting with a_3. The length is 3.

If there exists a dec sub-seq with length ≥ 3, we're done.

Case 2:

Assume that the lengths of all dec sub-seq are either 1 or 2. By P.P., there are 3 dec sub-seq with the same length, say 2. Let x_1, x_2, x_3 be the head of these 3 dec sub-seq. Then: $x_1 \leq x_2 \leq x_3$

Why:

Consider x_1 and x_2 and suppose $x_1 \geq x_2$. Since x_2 is the head of a dec sub-seq with length 2, appending it to x_1 would make the length 3, contradicting our assumption.

I.e.: $\underbrace{x_1, \underbrace{x_2, a}_{2}}_{3}$

The same logic applies for x_2 and x_3.

b) In any sub-seq of 17 nums, there is always an inc or dec sub-seq of 5 nums.

Soln:

Let the 17 nums be $a_1, a_2, \ldots, a_{17}$

Suppose we want to find a dec sub-seq.

Case 1:
Consider the max dec sub-seq starting from a_1. The length is 17.

We want to show that there exists a dec sub-seq with length ≥ 5.

Case 2:
Suppose that no dec sub-seq has length ≥ 5. Then, the possible lengths are 1, 2, 3, 4. By P.P., there must be 5 nums with the same length, say 4. Let these 5 nums be $x_1, x_2, \ldots, x_5$. Then $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$.
Why:
Consider x_1 and x_2. Suppose $x_1 > x_2$. Then, because x_2 is at the start of a dec sub-seq with length 4, adding x_1 to it will make it length 5, contradicting our assumption. The same logic applies to x_2, x_3, x_4, x_5.

c) Is the following statement correct:
"In any seq of 4 nums, there is always an inc or dec sub-seq of 3 nums."?

Soln:
No $n^2+1=4 \to n^2=3 \to n=\pm\sqrt{3}$

$n+1=\pm\sqrt{3}+1 \neq 3$

Counterpoint: 3, 4, 1, 2